Cocircuit Graphs and Efficient Orientation Reconstruction in Oriented Matroids

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Abstract

We consider the cocircuit graph $G_{\mathcal{M}}$ of an oriented matroid $\mathcal{M}$, which is the 1-skeleton of the cell complex formed by the span of the cocircuits of $\mathcal{M}$. As a result of Cordovil, Fukuda, and Guedes de Oliveira, the isomorphism class of $\mathcal{M}$ is not determined by $G_{\mathcal{M}}$, but it is determined if $\mathcal{M}$ is uniform and the vertices in $G_{\mathcal{M}}$ are paired if they are associated to negative cocircuits; furthermore the reorientation class of an oriented matroid $\mathcal{M}$ with $\text{rank}(\mathcal{M}) \geq 2$ is determined by $G_{\mathcal{M}}$ if every vertex in $G_{\mathcal{M}}$ is labeled by the zero support of the associated cocircuit. In this paper we show that the isomorphism class of a uniform oriented matroid is determined by the cocircuit graph, and we present polynomial algorithms which provide constructive proofs to all these results. Furthermore it is shown that the correctness of the input of the algorithms can be verified in polynomial time.

1 Introduction

The notion of oriented matroids (OMs) is a combinatorial abstraction of linear subspaces of the Euclidean space $\mathbb{R}^d$. The theory of OMs has applications and connections to a variety of different areas, including combinatorics, discrete and computational geometry, optimization, and graph theory; see e.g. Björner et al. [1]. Since OMs have several different representations, the choice of a representation and the translation from one into another representation are of practical interest; the present work discusses graph representations of OMs, focussing on algorithms and their complexity, and extends the work of Cordovil, Fukuda, and Guedes de Oliveira [4].

Consider a finite sphere arrangement $\mathcal{S} = \{S_e \mid e \in E\}$ in the Euclidean space $\mathbb{R}^{d+1}$, i.e. a collection of $(d-1)$-dimensional unit spheres on the d-dimensional unit sphere S^d, where every sphere S_e is oriented (i.e. has a $+$ side and a $-$ side). Figure 1 shows an example for $d = 2$ with $|E| = 4$ spheres. The sphere arrangement $\mathcal{S}$ induces a cell complex $\mathcal{W}$ on S^d. For every point x on S^d we define a *sign vector* $X \in \{+, 0, -\}^E$ by setting $X_e = 0$ if x is on S_e, otherwise $X_e = +$ (or $X_e = -$) if x is on the $+$ side (or $-$ side, respectively) of S_e; let $\mathcal{F}$ denote the set of all these sign vectors. Obviously there is a one-to-one correspondence between the cells in $\mathcal{W}$ and the sign vectors in $\mathcal{F}$. Let $\mathcal{C}$ denote the subset of $\mathcal{F}$ corresponding to the cells of dimension 0, then we call $X \in \mathcal{C}$ a *cocircuit* and the pair $\mathcal{M} = (E, \mathcal{C})$ *a linear OM*. Analogously, for $\mathcal{S}$ being an arrangement of pseudospheres [1, 5] we call $\mathcal{M} = (E, \mathcal{C})$ *an OM*. The 1-skeleton of $\mathcal{W}$ is a graph G which is defined by the OM $\mathcal{M}$ (see Section 2) and what we call *the cocircuit graph* $G_{\mathcal{M}}$ *of* $\mathcal{M}$. A graph G is *a cocircuit graph* if $G = G_{\mathcal{M}}$ for some OM $\mathcal{M}$. In Figure 1 the cocircuits are $A = (0, +, +, 0)$, $B = (0, +, 0, +)$, $C = (0, 0, -, +)$, $D = (+, 0, 0, 0)$, and their negatives $-A$, $-B$,

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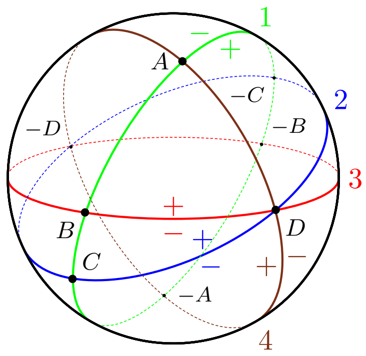

Figure 1: A Sphere Arrangement on S^2

$-C$, $-D$, and these eight cocircuits correspond to the vertices $v_A, \dots, v_{-D}$ of the cocircuit graph as it is depicted on the surface of S^2.

Compared to the set of sign vectors $\mathcal{F}$ of a cell complex $\mathcal{W}$, the cocircuit graph is a compact and simple structure. It is a natural question, whether the cocircuit graph of an OM $\mathcal{M}$ determines the cell complex $\mathcal{W}$. In the OM language, this question amounts to: does $G_{\mathcal{M}}$ determine the isomorphism class $IC(\mathcal{M})$ of $\mathcal{M}$? Remark that $G_{\mathcal{M}}$ is a graph with no additional information. It might be easier to determine $IC(\mathcal{M})$ from $G_{\mathcal{M}}$ and for example with the vertices paired as they are associated to negative cocircuits. Such additional information will be added to $G_{\mathcal{M}}$ in form of a *label* which is a mapping defined on the vertex set of $G_{\mathcal{M}}$: an *OM-label* $\mathcal{L}$ maps each vertex to the associated cocircuit (e.g. $\mathcal{L}(v_A) = (0, +, +, 0)$, $\mathcal{L}(v_B) = (0, +, 0, +)$), an *M-label (matroid label)* L maps each vertex to the associated hyperplane of the underlying matroid (e.g. $L(v_A) = \{1, 4\}$, $L(v_B) = \{1, 3\}$), and an *AP-label (antipodal label)* maps each vertex to its *antipodal* (e.g. v_A is mapped to v_{-A}, v_B to v_{-B}). Correspondingly, the graph will be called *OM-labeled*, *M-labeled*, and *AP-labeled*. In order to give an answer to the above question whether $IC(\mathcal{M})$ is determined by $G_{\mathcal{M}}$, we decompose the problem into the following two problems:

M-Labeling Problem: *Given a cocircuit graph G, find an M-label of G.*

OM-Labeling Problem: *Given a cocircuit graph G with M-label L, find an OM-label $\mathcal{L}$ of G such that L is the M-label of G induced by $\mathcal{L}$ (i.e. $L(v)$ is the zero support of $\mathcal{L}(v)$ for each vertex v).*

Cordovil, Fukuda, and Guedes de Oliveira [4] showed that in general the M-labeling problem has solutions that are not isomorphic to each other, i.e. the cocircuit graph of an OM $\mathcal{M}$ does not determine $IC(\mathcal{M})$. In contrast, if $\mathcal{M}$ is uniform and $G_{\mathcal{M}}$ is AP-labeled, then $IC(\mathcal{M})$ is uniquely determined; we discuss this case in Section 3 and present a polynomial algorithm which computes an M-label. In Section 4 we strengthen the result and show that the isomorphism class of a uniform OM $\mathcal{M}$ is determined by $G_{\mathcal{M}}$ (without AP-label), and we present a polynomial algorithm that solves the M-labeling problem for uniform OMs.

Concerning the OM-labeling problem it was also proved in [4] that for any OM $\mathcal{M}$ with $\text{rank}(\mathcal{M}) \geq 2$ the M-labeled cocircuit graph $G_{\mathcal{M}}$ determines the reorientation class $OC(\mathcal{M})$. We will discuss this in Section 5 and present a simple polynomial algorithm that finds an OM-label from an M-labeled cocircuit graph.

The following two problems are naturally related to the labeling problems:

Characterization Problem: *Decide whether a given graph (without or with label) is a cocircuit graph.*

Covector Construction Problem: *Given the set $\mathcal{C}$ of cocircuits of an OM $\mathcal{M}$, construct the set $\mathcal{F}$ of covectors of $\mathcal{M}$.*

We discuss in Section 6 how the correctness of the input of our algorithms can be checked in polynomial time. This solves the characterization problem for cocircuit graphs of uniform OMs and for M-labeled cocircuit graphs. When $\text{rank}(\mathcal{M}) = 3$, the cocircuit graph $G_{\mathcal{M}}$ is planar and has a unique dual, known as the tope graph of $\mathcal{M}$; there is a polynomial characterization of tope graphs for $\text{rank}(\mathcal{M}) = 3$ (see Fukuda-Handa [6]), hence also for rank 3 cocircuit graphs. For the general case there is no polynomial characterization known.

The covector construction problem will be discussed in Section 7. We present an algorithm which solves the problem in polynomial time measured in input and output, as $|\mathcal{F}|$ can be exponential in $|\mathcal{C}|$. The solution of the covector construction problem completes the reconstruction of an OM face lattice from the cocircuit graph.

Our algorithmic solutions extend the work of [4], and we also simplify some of the proofs given there. The present work is also related to Perles's conjecture which says that the 1-skeleton of a simple d-dimensional polytope determines its face lattice; this conjecture was first proved by Blind and Mani-Levitska [2] and then constructively by Kalai [9]. If an OM is linear, the cell complex $\mathcal{W}$ formed by $\mathcal{F}$ is isomorphic to the face lattice of the dual of a zonotope, i.e. the present work extends the discussion of Perles's conjecture to a class of non-simple polytopes. Joswig [8] conjectured that every cubical polytope can be reconstructed from its dual graph; our result proves this conjecture for the special case of cubical zonotopes up to graph isomorphism. In other words, the face lattice of every cubical zonotope is uniquely determined by its dual graph up to isomorphism. Mnëv [10] proved that it is $\mathcal{NP}$-hard to decide whether a given OM is linear or not (for a simpler proof see also Shor [12]). Since the cocircuit graph of an OM can be constructed in polynomial time from the cocircuits, it is also $\mathcal{NP}$-hard to decide whether a given cocircuit graph is the cocircuit graph of a linear OM, i.e. there is no polynomial characterization of the cocircuit graphs of linear OMs unless $\mathcal{P} = \mathcal{NP}$. For reconstruction of an OM from the orientation classes of one-element deletions see Roudneff [11], for other combinatorial characterizations of OMs see Cordovil and Fukuda [3] and Hochstättler [7].

2 Definitions and Notations

We present the definitions and the notations used in this paper, as far as not introduced in Section 1. Some notions are defined again, extending their former meaning in the setting of the sphere model to the axiomatic of OMs as presented in the following.

The *zero support* of a sign vector $X \in \{+,0,-\}^E$ is the set $X^0 := \{e \in E \mid X_e = 0\}$, and *the negative* $-X$ *of* X is defined by $(-X)_e := -X_e$ for $e \in E$. For two sign vectors $X, Y \in \{+,0,-\}^E$ we say that X *conforms to* Y (denoted by $X \preceq Y$) if $X_e \neq 0$ implies $X_e = Y_e$. The *composition of* X *and* Y (denoted by $X \circ Y$) is the sign vector W with $W_e = Y_e$ for $e \in X^0$ and $W_e = X_e$ otherwise.

An OM $\mathcal{M}$ is a pair $(E, \mathcal{C})$ of a finite set E and a set $\mathcal{C} \subseteq \{+,0,-\}^E$ of sign vectors (called *cocircuits*) for which the OM cocircuit axioms (C1) to (C4) are valid:

(C1) $\mathbf{0} \notin \mathcal{C}$.

(C2) $V \in \mathcal{C} \Rightarrow -V \in \mathcal{C}$.

(C3) $V, W \in \mathcal{C}$, $V^0 \subseteq W^0 \Rightarrow V = W$ or $V = -W$.

(C4) For all $V, W \in \mathcal{C}$, $V \neq -W$, $e \in E$ such that $V_e = -W_e \neq 0$ there exists $X \in \mathcal{C}$ such that $X_e = 0$ and for all $f \in E$ is $X_f \in \{V_f, W_f, 0\}$.

It is not difficult to see that these OM cocircuit axioms hold for any OM $(E, \mathcal{C})$ as defined by pseudosphere arrangements $\mathcal{S}$ in Section 1; furthermore the Topological Representation Theorem of Folkman and Lawrence [5] states that every OM as defined by the above axioms has a pseudosphere representation.

A composition of cocircuits is called a *covector* (and in addition we also call the zero vector $\mathbf{0} \in \{+, 0, -\}^E$ a covector). The set $\mathcal{F}$ of all covectors ordered by the conformal relation $\preceq$, together with an additional artificial greatest element 1, forms a lattice $\hat{\mathcal{F}}$ which has the Jordan-Dedekind property. The *rank* of a covector X is defined as the height of X in $\hat{\mathcal{F}}$, and we define the rank of $\mathcal{M}$ by $\operatorname{rank}(\mathcal{M}) := \max_{X \in \mathcal{F}} \operatorname{rank}(X)$. We denote by $\underline{\mathcal{F}} := \{X^0 | X \in \mathcal{F}\}$ the flats of the *underlying matroid* $\underline{\mathcal{M}}$ of $\mathcal{M}$. The zero supports of cocircuits are called *hyperplanes*, the zero supports of the rank 2 elements in $\hat{\mathcal{F}}$ are called *colines*. An OM $\mathcal{M}$ is called *uniform* if the set of hyperplanes is the set of all $(\operatorname{rank}(\mathcal{M}) - 1)$-subsets of E.

A *graph* $G = (V(G), E(G))$ is a pair of a finite set of *vertices* $V(G)$ and a set of *edges* $E(G)$ that are represented as unordered pairs of vertices, and in this paper we identify any two graphs that are isomorphic. *The cocircuit graph of an OM* $\mathcal{M} = (E, \mathcal{C})$ is a graph G with exactly $|\mathcal{C}|$ vertices that can be associated by a bijection $\mathcal{L} : V(G) \to \mathcal{C}$ to the cocircuits of $\mathcal{M}$ such that $\{v, w\}$ is an edge in $E(G)$ if and only if for $V := \mathcal{L}(v)$ and $W := \mathcal{L}(w)$ holds: $V \circ W = W \circ V$ and V and W are the only cocircuits conforming to $V \circ W$. For any edge $\{v, w\} \in E(G)$ is $U := (\mathcal{L}(v) \circ \mathcal{L}(w))^0$ a coline, and we say that $\{v, w\}$ is *an edge on coline* U.

A *label of a graph* G is a mapping L defined on the vertex set $V(G)$, and we call $L(v)$ *the label of* $v \in V(G)$. For a graph G and an OM $\mathcal{M}$ we call a label $\mathcal{L}$ of G *the OM-label of* G *w.r.t.* $\mathcal{M}$ if $G = G_{\mathcal{M}}$ and every vertex v is labeled by the cocircuit associated to v; we call a label $\mathcal{L}$ of a graph G *an OM-label of* G if $\mathcal{L}$ is the OM-label of G w.r.t. some OM. If we omit orientations, we obtain a labeling by the underlying matroid: For an OM-label $\mathcal{L}$ of a graph G we call a label L of G *the M-label of* G *induced by* $\mathcal{L}$ if every vertex v is labeled by the zero support $\mathcal{L}(v)^0$; we call a label L of a graph G *an M-label of* G if L is the M-label of G induced by some OM-label of G. The labels of two vertices given by an M-label are the same if and only if they correspond to negative cocircuits; we call such vertices *antipodals* or *an antipodal pair*, and define: For an M-label L of a graph G we call a label of G *the AP-label of* G *induced by* L if every vertex v is mapped to the *antipodal* $\overline{v}$ *of* v which is the unique vertex $\overline{v} \in V(G) \setminus \{v\}$ such that $L(v) = L(\overline{v})$; for a graph G we call a label of G *an AP-label of* G if it is the AP-label of G induced by some M-label of G.

3 M-Labeling from AP-Label

We discuss in this section the M-labeling problem where the given graph G is the cocircuit graph of some uniform OM and where an AP-label A of G is given. W.l.o.g. we will not consider M-labels of G that are not induced by a uniform OM. We present a polynomial algorithm MLabelFromAPLabel which computes an M-label L of G such that A is the AP-label of G induced by L. By this we extend the result of [4] which states that such an M-label is unique up to isomorphism on the ground set, which is the union of the vertex labels. Remark that for OMs of rank 0 or 1 the M-labeling problem is trivial, and we can assume for the following that $\operatorname{rank}(\mathcal{M}) \geq 2$. Remark that for the algorithm MLabelFromAPLabel no information like $\mathcal{M}$, E, or $\operatorname{rank}(\mathcal{M})$

is given; we will only use G, the given AP-labeling $A : v \mapsto \overline{v}$, and the information that $\mathcal{M}$ is uniform. This uniformity implies many structural properties:

Lemma 1 *Let $\mathcal{M} = (E, \mathcal{C})$ be a uniform OM with $\ell := |E|$ and $r := \operatorname{rank}(\mathcal{M}) \geq 2$. Then:*

(i) *Every subset of $r-1$ elements is a hyperplane, and every subset of $r-2$ elements is a coline.*

(ii) *For any coline $U \subseteq E$, the edges on U form a cycle in $G_{\mathcal{M}}$ of length $2 \cdot (\ell - r + 2)$. We call this cycle the coline cycle of U.*

(iii) *The coline cycles of any two different colines U_1 and U_2 have a common vertex if and only if $|U_1 \setminus U_2| = 1$.*

Proof: (i) follows directly from the uniformity of $\mathcal{M}$. If U is a coline, then the contraction minor $\mathcal{M}/U$ is a uniform OM of rank 2 on a ground set of cardinality $\ell - (r-2)$, which implies (ii). From (i) and (ii) follows that a vertex v is on the cycle of a coline U if and only if the hyperplane associated to v is $U \cup \{e\}$ for some $e \in E \setminus U$, which implies (iii). ■

Let $\mathcal{M} = (E, \mathcal{C})$ be a uniform OM, L the M-label induced by the OM-label of $G_{\mathcal{M}}$ w.r.t. $\mathcal{M}$, and $v_0 \in V(G_{\mathcal{M}})$ an arbitrary vertex. For a coline $U \subseteq E$ we call $|U \setminus L(v_0)|$ *the distance of U to v_0* and also *the distance of the coline cycle of U to v_0*. Lemma 1 implies that the coline cycles of distance 0 are the coline cycles through v_0, the coline cycles of distance 1 are those which intersect a coline cycle of distance 0 but do not meet v_0; inductively the coline cycles of distance $k+1$ are exactly those that intersect at least one coline cycle of distance k but which are not of distance k. Hence the distance of a coline is also defined by the cocircuit graph and the coline cycles (i.e. without hyperplanes and colines). The following lemma states an important property of coline cycles:

Lemma 2 *Let $\mathcal{M} = (E, \mathcal{C})$ be a uniform OM with $\ell := |E|$ and $r := \operatorname{rank}(\mathcal{M}) \geq 2$, let p be a path $v = v_0, v_1, v_2, \ldots, v_{t-1}, v_t = \overline{v}$ in $G_{\mathcal{M}}$ connecting an antipodal pair $(v, \overline{v})$. Then: p is a shortest path in $G_{\mathcal{M}}$ from v to $\overline{v}$ if and only if $t = \ell - r + 2$, and then there exists a coline $U \subseteq E$ such that $\{v_{i-1}, v_i\}$ is an edge on the coline cycle of U for all $i \in \{1, \ldots, t\}$.*

Proof: Let L be the M-label induced by the OM-label of $G_{\mathcal{M}}$ w.r.t. $\mathcal{M}$. Obviously there are $2 \cdot (r-1)$ different paths from v to $\overline{v}$ of length $\ell - r + 2$ that are defined by the $r-1$ coline cycles through v and $\overline{v}$. On the other hand, let p be a path from v to $\overline{v}$, and let $J \subseteq E$ be the set of elements that belong to some but not all labels of the vertices v_i on p. Since by uniformity $|L(v_{i-1}) \setminus L(v_i)| = 1$ for each edge $\{v_{i-1}, v_i\}$ on p, $L(v) = L(\overline{v})$ implies that the cardinality $|J|$ is a lower bound for the length of p. Certainly $E \setminus L(v) \subseteq J$, and if p does not follow only one coline, then $|L(v) \cap J| \geq 2$, i.e. then the length of p is at least $|E \setminus L(v)| + 2 = \ell - r + 3$. ■

The algorithmic idea is first to detect the coline cycles of the cocircuit graph with an algorithm ListColineCycles with input and output as specified in Table 1, and then to use these coline cycles to construct an M-label with an algorithm MLabelFromColineCycles (see Table 2); the two steps could be done in parallel, but for clarity and since there is no loss w.r.t. complexity we present the algorithm MLabelFromAPLabel divided into this two parts (cf. Table 3).

It is not difficult to design an algorithm ListColineCycles as specified in Table 1 which runs in time of at most $O(nm)$, where $n := |V(G)|$ and $m := |E(G)|$; it is sufficient to visit all antipodal pairs with increasing coline distance to v_0, to determine for each pair $(v, \overline{v})$ the $2(r-1)$ shortest paths between v and $\overline{v}$, and to combine two such paths to a coline cycle when they contain antipodal vertices (cf. Lemma 2).

Input: A cocircuit graph G with AP-label A, and $v_0 \in V(G)$. *Output:* A list S of all coline cycles of G such that every coline cycle $s \in S$ is given as a list of the vertices on s in an order as they are adjacent on s, and such that S is ordered with increasing coline distance to vertex v_0, and among the coline cycles of distance 1 those come first which intersect the first coline cycle in S.

Table 1: Input and Output Specification of ListColineCycles

Input: A list S as specified as output of ListColineCycles. *Output:* An M-label L of the graph G given by S.

Table 2: Input and Output Specification of MLabelFromColineCycles

The key ideas of algorithm MLabelFromColineCycles are an initialization of the labels as far as the freedom of isomorphism allows, and then the propagation of the labels observing necessary conditions; finally the coline cycle connectivity will be used to prove that the construction of the M-label has been complete. The necessary conditions for propagation and the coline cycle connectivity are stated in the following lemma:

Lemma 3 *Consider the cocircuit graph G of a uniform OM, an M-label L of G, and the coline cycles in G given by L.*

(i) *If v and w are vertices on a common coline cycle s and not antipodals, then the intersection $L(s)$ of all labels of vertices on s is equal to $L(v) \cap L(w)$.*

(ii) *If v is a vertex on two different coline cycles s_1, s_2, then $L(v) = L(s_1) \cup L(s_2)$.*

(iii) *On a coline cycle of distance $k \geq 1$ to v_0 there are exactly $2 \cdot (k+1)$ vertices that are on at least one coline cycle of distance $k-1$; every of these vertices is on exactly k coline cycles of distance $k-1$.*

Proof: All claims follow from the uniformity of $\mathcal{M}$; see also Lemma 1. ■

Given an M-label L, we call for a coline cycle s the set $L(s)$ as introduced in Lemma 3 *the label of s*. We discuss now initialization and propagation of the labels in the construction of an M-label by algorithm MLabelFromColineCycles. Given is a set S from algorithm ListColineCycles.

Initialization. We can easily determine $r := \text{rank}(\mathcal{M})$ and $\ell := |E|$ from S, since every vertex appears on exactly $r-1$ coline cycles and every coline cycle has length $2 \cdot (\ell - r + 2)$. Using the freedom of isomorphism we initialize $L(v_0) := \{1, \dots, r-1\}$, and of course $L(\overline{v_0}) := L(v_0)$, and the labels of the remaining $2 \cdot (\ell - r + 1)$ vertices on the first coline cycle in S are set to $\{1, \dots, r-2\} \cup \{j\}$ for $j \in \{r, \dots \ell\}$, where antipodal vertices take the same label. Hence the label of the first coline cycle in S is set to $\{1, \dots r-2\}$; we are still free to initialize the labels of the remaining coline cycles s_i of distance 0 (i.e. the coline cycles at a position $i \in \{2 \dots, r-1\}$ in S) by $L(s_i) := \{1, \dots r-1\} \setminus \{i-1\}$ (i.e. we initialize the label of every vertex v on s_i that is different from v_0 and $\overline{v_0}$ by $L(v) := L(s_i)$).

Propagation. In the order of list S, i.e. with increasing distance to vertex v_0, and starting with the first coline cycle of distance 1 (this coline cycle is at position r in S) we do the following for every coline cycle s:

1. We determine the label $L(s)$ as follows:
   - If s is of distance 1 and intersects the first coline cycle in S, the only two distinct labels already initialized on s have the form $\{1,\dots,r-2\}\cup\{j\}$ for $j\in\{r,\dots\ell\}$ and $L(s_i)=\{1,\dots r-1\}\setminus\{i-1\}$ for $i\in\{2\dots,r-1\}$; the label must then be $L(s):=\{1,\dots,r-2\}\setminus\{i-1\}\cup\{j\}$.
   - If s is of distance 1 and does not intersect the first coline cycle in S, then there are two distinct labels already initialized on s which have the form $\{1,\dots,r-1\}\setminus\{i_1-1\}\cup\{j\}$ and $\{1,\dots,r-1\}\setminus\{i_2-1\}\cup\{j\}$ for $i_1,i_2\in\{2,\dots r-1\}$ with $i_1\neq i_2$ and $j\in\{r,\dots\ell\}$; the label must then be their intersection, i.e. $L(s):=\{1,\dots,r-1\}\setminus\{i_1-1\}\setminus\{i_2-1\}\cup\{j\}$.
   - If s is of distance $k\geq 2$, then we choose any two among the $k+1$ labels already initialized on s; these labels are already determined by $k\geq 2$ vertices of distance $k-1$, hence $L(s)$ is equal to the intersection of these two labels.
2. We add $L(s)$ to $L(v)$ for every vertex v on the coline cycle, i.e. $L(v):=L(v)\cup L(s)$; for the first time we set $L(v):=L(s)$, and after the next change will $L(v)$ be a $(r-1)$-subset of E, i.e. $L(v)$ is then a complete vertex label and will not be changed further.

Initialization and propagation describe the algorithm MLabelFromColineCycles, hence also the algorithm MLabelFromAPLabel is now complete (see Table 3).

Input: A cocircuit graph G with AP-label A.
Output: An M-label L of G such that A is the AP-label of G induced by L.

begin MLabelFromAPLabel(G, A);
  Choose any vertex $v_0\in V(G)$;
  S := ListColineCycles(G, A, v_0);
  return MLabelFromColineCycles(S)
end MLabelFromAPLabel.

Table 3: Algorithm MLabelFromAPLabel

Theorem 4 *If G is the cocircuit graph of a uniform OM $\mathcal{M}$ with $\operatorname{rank}(\mathcal{M})\geq 2$ and A an AP-label of G, then the algorithm* MLabelFromAPLabel *terminates with correct output in time $O(nm)$, where $n:=|V(G)|$ and $m:=|E(G)|$. The M-label L constructed by* MLabelFromAPLabel *is unique up to isomorphism on the ground set.*

Proof: Let $\mathcal{M}=(E,\mathcal{C})$ be a uniform OM with $\ell:=|E|$ and $r:=\operatorname{rank}(\mathcal{M})\geq 2$, and in addition we set $u:=\binom{\ell}{r-2}$ for the number of colines. We have already seen that with input $G=G_{\mathcal{M}}$ the algorithm determines all labels correctly and – up to isomorphism – uniquely because of the properties stated in Lemma 3 (remark that in the special case $\operatorname{rank}(\mathcal{M})=2$, the labels are complete after initialization of the first coline cycle). The complexity of ListColineCycles was stated to be $O(nm)$, and we will show that the complexity of MLabelFromColineCycles is of order $O(m)+O(r\cdot u)$, which is also at most $O(nm)$ because $\ell\geq r$ implies $n=2\binom{\ell}{r-1}\geq 2\binom{r}{r-1}=2r$ and $m=2u(\ell-r+2)\geq 4u$, hence $nm\geq 8ru$. In MLabelFromColineCycles we visit every vertex in every coline cycle not more than some constant number of times (from there $O(m)$ operations). We modify the label of every vertex at most twice, and since we can keep labels sorted we need $O(r)$ operations for one modification, which leads to a total number of $O(nr) = O(m)$ operations for all label modifications. Finally we need for every of the u coline cycles $O(r)$ computations to find its label. ■

4 M-Labeling without AP-Label

In this section we dicuss how to solve the M-labeling problem for a cocircuit graph $G_{\mathcal{M}}$ of a uniform $\mathcal{M}$ without AP-label, by this strengthening the result of the previous section. Again we will not consider M-labels that are not induced by a uniform OM. We first discuss how to construct an M-label when the labels of only two antipodal pairs on a common coline are given:

Theorem 5 *If G is the cocircuit graph of a uniform OM $\mathcal{M}$ and there are two different antipodal pairs labeled in G which are known to be on a common coline cycle, then one can construct an M-label L of G in time $O(nm)$, where $n := |V(G)|$ and $m := |E(G)|$, and the AP-label of G induced by L is uniquely determined by G and the two given antipodal pairs.*

Proof: Let $v, \overline{v}$ and $w, \overline{w}$ be two different antipodal pairs in G that are on a common coline cycle s. As for the label construction in the previous section, $r := \text{rank}(\mathcal{M})$ and the cardinality ℓ of the ground set of $\mathcal{M}$ can be easily found from the degree $2 \cdot (r-1)$ of a vertex and the distance $\ell - r + 2$ of an antipodal pair. Let E be a set of cardinality ℓ. We know that for any M-label L of G with ground set E the vertex labels $L(v) = L(\overline{v})$ and $L(w) = L(\overline{w})$ are $(r-1)$-subsets of E and $L(s) = L(v) \cap L(w)$ is a $(r-2)$-subset of E, hence $L(v) = L(s) \cup \{e_v\}$ and $L(w) = L(s) \cup \{e_w\}$ for $e_v, e_w \in E \setminus L(s)$, where $e_v \neq e_w$. There are $2 \cdot (r-1)$ shortest paths between v and $\overline{v}$, each corresponding to one half of a coline cycle (see Lemma 2), and the same holds for w and $\overline{w}$; we have to detect which paths belong to the same coline cycle. It is easy to find the shortest paths belonging to the coline cycle s which contains the given antipodal pairs. Two shortest paths not belonging to s, say p_1 between v and $\overline{v}$ and p_2 between w and $\overline{w}$, belong to coline cycles s_1 and s_2 with labels $L(s_1) = L(v) \setminus \{e_1\}$ and $L(s_2) = L(w) \setminus \{e_2\}$ for some $e_1, e_2 \in L(s)$, and since $L(s_1) \setminus L(s_2) = \{e_v, e_2\} \setminus \{e_1\}$, the paths p_1 and p_2 have a common vertex (an *intersection vertex*) if and only if $e_1 = e_2$ (cf. Lemma 1 (iii)); the label of the intersection vertex is $L(s) \cup \{e_v, e_w\} \setminus \{e_1\}$. It is easy to see that there are exactly $2 \cdot (r-2)$ intersection vertices (namely $r-2$ antipodal pairs) with labels $L(s) \cup \{e_v, e_w\} \setminus \{e_i\}$ for $e_i \in L(s)$, and hence any two intersection vertices are on a common coline cycle with a label of the form $L(s) \cup \{e_v, e_w\} \setminus \{e_i, e_j\}$. Therefore the distance of two intersection vertices in G is less or equal to $\ell - r + 2$ with equality if and only if they are antipodals; by this we can identify shortest paths belonging to the same coline cycle. Hence we can determine all coline cycles of distance 0 to v and with the same technique for the rest of G, extending the labeling as in the algorithm MLabelFromColineCycles. Also the complexity discussion is similar to the discussion above, it is sufficient to count all costs for computing shortest paths and identifying antipodal intersection vertices correctly (for every of the n vertices there are total costs of $O(m)$). ■

Theorem 5 implies that there is an algorithm which solves the M-labeling problem for a given cocircuit graph G of a uniform OM $\mathcal{M} = (E, \mathcal{C})$ without AP-label in time $O(n^3 m \ell^2)$, where $n := |V(G)|$, $m := |E(G)|$, $\ell := |E|$: For a choice of two pairs of vertices $(v, \overline{v})$ and $(w, \overline{w})$ from G, we construct a label L of G as in the proof of Theorem 5 (this might fail, then $(v, \overline{v})$ and $(w, \overline{w})$ are not two antipodal pairs); if L is an M-label of G (we can check this in time $O(n^3 \ell^2)$, see Theorem 13), we stop, otherwise $(v, \overline{v})$ and $(w, \overline{w})$ are not two antipodal pairs and we start over with other pairs. Obviously it is sufficient to check pairs where $\{v,w\}$ and $\{\overline{v},\overline{w}\}$ are edges in G and one edge is fix, i.e. there are at most $O(m)$ pairs to check.

It remains to discuss whether the M-labels of a graph G that is the cocircuit graph of a uniform OM are all isomorphic, i.e. whether for any two M-labels $L : V(G) \to 2^E$ and $\tilde{L} : V(G) \to 2^{\tilde{E}}$ there exists a bijection $\phi : E \to \tilde{E}$ such that $\tilde{L} = \phi L$. We will prove this up to graph automorphism in Theorem 7, using Theorem 5 and the following Lemma 6:

Lemma 6 *Let G be the cocircuit graph of a uniform OM $\mathcal{M} = (E,\mathcal{C})$ with $\operatorname{rank}(\mathcal{M}) = 2$ or $\operatorname{rank}(\mathcal{M}) = 3$, and $v,w \in V(G)$. The distance from v to w in G is at most $|E| - \operatorname{rank}(\mathcal{M}) + 2$ with equality if and only if v and w are antipodals.*

Proof: Let $\mathcal{L}$ be the OM-label of G w.r.t. $\mathcal{M}$, and set $V := \mathcal{L}(v)$ and $W := \mathcal{L}(w)$. We assume that V and W are not on a common coline and therefore $\operatorname{rank}(\mathcal{M}) = 3$, otherwise the claim is obviously correct. W.l.o.g. we assume that $E = \{1,\ldots,\ell\}$, $V^0 = \{1,2\}$, $3 \in W^0$, and $W_1 = W_2 = V_3 = +$. We consider for $i \in I := \{1,2,3\}$ the colines $\{i\}$ and their coline cycles s_i. For $i \in I$ let X^i be the cocircuit defined by $X^i_i = +$ and $X^i_j = 0$ for $j \in I \setminus \{i\}$, then the vertex x_i corresponding to X^i is on the intersection of s_j and s_k for $\{j,k\} = I \setminus \{i\}$ (especially $v = x_3$). Denote by p_i the shorter of the two paths on s_i between x_j and x_k, where $\{j,k\} = I \setminus \{i\}$. Then the union p of the paths p_1, p_2, p_3 forms a cycle in G, and a vertex $y \in V(G)$ is on p if and only if $\mathcal{L}(y)_I \in \{0,+\}^I \setminus (\{0\}^I \cup \{+\}^I)$. As v and w are on p, it is sufficient to prove that the length of p is less than $2(\ell-1)$. We show that there are at most $2(\ell-3)$ vertices y on p different from x_1, x_2, and x_3: Such a vertex y is characterized by $\mathcal{L}(y)_e = 0$ for some $e \in E \setminus I$ and $\mathcal{L}(y)_i = 0$ for some $i \in I$, and then $\mathcal{L}(y)_j = +$, $\mathcal{L}(y)_k = +$ for $\{j,k\} = I \setminus \{i\}$. Assume that for some $e \in E \setminus I$ there exist all three vertices, i.e. there exist three cocircuits in $\mathcal{C}$ whose signs corresponding to $(1,2,3,e)$ are $(0,+,+,0)$, $(+,0,+,0)$, and $(+,+,0,0)$; then the cocircuit axiom (C4) applied to the first and the negative of the second implies a contradiction to axiom (C3) for the third cocircuit. Therefore there exist for every $e \in E \setminus I$ at most two vertices y on p with $\mathcal{L}(y)_e = 0$. ■

Theorem 7 *Let G be the cocircuit graph of a uniform OM $\mathcal{M}$ and L and $\tilde{L}$ M-labels of G. Then there exists a graph automorphism $g \in \operatorname{Aut}(G)$ such that Lg and $\tilde{L}$ are isomorphic.*

Proof: Let L and $\tilde{L}$ be M-labels of G, and denote the induced AP-labels by A and $\tilde{A}$, respectively. Remark that $A^{-1} = A \in \operatorname{Aut}(G)$ and $\tilde{A}^{-1} = \tilde{A} \in \operatorname{Aut}(G)$. Since for any $g \in \operatorname{Aut}(G)$ the AP-label induced by Lg is $g^{-1}Ag$ and because of Theorem 4, it is sufficient to find $g \in \operatorname{Aut}(G)$ such that $g^{-1}Ag = \tilde{A}$. As $\operatorname{Aut}(G)$ is finite, the order of $\tilde{A}A \in \operatorname{Aut}(G)$ is finite. If the order of $\tilde{A}A$ is odd, say $2k+1$ for a nonnegative integer k, then $g := (\tilde{A}A)^k$ is sufficient. We will show that the order of $\tilde{A}A$ cannot be even.

We show that $(\tilde{A}A)^2 = 1$ implies $\tilde{A}A = 1$ (hence the order of $\tilde{A}A$ cannot be 2). Let E denote the ground set of L, and as usual $\ell := |E|$ and $r := \operatorname{rank}(\mathcal{M})$. Assume $(\tilde{A}A)^2 = 1$, then the AP-labels induced by $L\tilde{A}$ and L are equal, so by Theorem 4 $L\tilde{A}$ and L are isomorphic, i.e. there exists a permutation $\pi \in S_E$ such that $\pi L = L\tilde{A}$. As $\pi\pi L = \pi L\tilde{A} = L\tilde{A}\tilde{A} = L$ implies $\pi^2 = 1$, the orbits of π must all have order 1 or 2, so we can choose a union $U \subseteq E$ of these orbits with $|U| = r-2$ or $|U| = r-3$. Consider the subgraph G_U of G induced by the vertex set $V(G_U) := \{v \in V(G) \mid U \subseteq L(v)\}$. Remark that $V(G_U)$ is closed under A by definition and also closed under $\tilde{A}$ because of $L\tilde{A} = \pi L$ and $\pi(U) = U$. G_U is the cocircuit graph of a uniform OM contraction minor with rank $r' := r - |U| \in \{2,3\}$ and $\ell' := \ell - |U|$ elements in the ground set, so Lemma 6 implies that for every vertex $v \in V(G_U)$ there is a unique vertex $\overline{v} \in V(G_U)$ such that the distance in G_U from v to $\overline{v}$ is at least $\ell' - r' + 2 = \ell - r + 2$. On the other hand $\ell - r + 2$ is the distance in G between a vertex v and $A(v)$ (and also between v and $\tilde{A}(v)$), and the distance in the subgraph G_U cannot be smaller. Therefore $A(v) = \tilde{A}(v) = \overline{v}$ for $v \in V(G_U)$, so by Theorem 5 follows $A = \tilde{A}$.

Assume that the order of $\tilde{A}A$ is $2k$ for an integer $k > 1$. If $k = 2k'$ set $\hat{L} := L(\tilde{A}A)^{k'-1}\tilde{A}$, if $k = 2k'+1$ set $\hat{L} := \tilde{L}(A\tilde{A})^{k'}$. Let $\hat{A}$ denote the AP-label induced by the M-label $\hat{L}$, then in either case $\hat{A}A = (\tilde{A}A)^k$, hence $(\hat{A}A)^2 = 1$. Thus by the previous case $(\tilde{A}A)^k = \hat{A}A = 1$, contradicting the assumption that the order of $\tilde{A}A$ is $2k$. ∎

5 OM-Labeling from M-Label

We consider the OM-labeling problem for an M-labeled cocircuit graph G of some OM $\mathcal{M}$. Remark that for OMs of rank 0 or 1 the problem is trivial, hence we will assume in the following that $\text{rank}(\mathcal{M}) \geq 2$, because then the ground set E of $\mathcal{M}$ is determined by the given M-label L as the union of all vertex labels $L(v)$. We extend the work of [4] as we slightly simplify the proof for the claim that the reorientation class $OC(\mathcal{M})$ is determined by G and L and present a simple polynomial algorithm OMLabelFromMLabel that solves the problem. The key argument is given by the following proposition:

Proposition 8 *Let $\mathcal{L}$ be an OM-label of G and L the M-label of G induced by $\mathcal{L}$, and for any $e \in E$ let be $G(e)$ the subgraph of G induced by the vertices v with $e \notin L(v)$. Then there are exactly two connected components of $G(e)$, and any two vertices v and w belong to the same connected component if and only if $\mathcal{L}(v)_e = \mathcal{L}(w)_e \neq 0$.*

A proof of Proposition 8 was given in [4], in the proof of Theorem 2.3. Besides some slight simplifications, our proof is based on the same ideas. The following property of hyperplanes in a matroid is needed:

Lemma 9 *Let (E, M) be a matroid of rank $r \geq 2$ with ground set E and set M of flats. For any two different hyperplanes $H, H^* \in M$ such that $H \cap H^*$ is not a coline and any $e \in E \setminus (H \cup H^*)$ there exists a hyperplane $H' \in M$ such that*

(i) $e \notin H'$,

(ii) $H \cap H'$ *is a coline, and*

(iii) $H \cap H^* \subsetneqq H' \cap H^*$.

Proof: Let U be a coline such that $H \cap H^* \subsetneqq U \subsetneqq H$, and let I be the intersection of all hyperplanes containing U and some $e^* \in H^* \setminus U$. If $U \subsetneqq I$, then I is a hyperplane and every hyperplane containing U and some $e^* \in H^* \setminus U$ is equal to I, by this $U \subseteq I = H^*$ and $U \subseteq H \cap H^*$, a contradiction. We conclude $U = I$, and since $e \notin U$ there exists a hyperplane H' containing U and some $e^* \in H^* \setminus U$ such that $e \notin H'$. The claim follows for H', observing that $e^* \in H' \setminus H$ (remark $e^* \notin U \supseteq H \cap H^*$, so $e^* \notin H$) and $H \cap H' = U$. ∎

Proof of Proposition 8: Let v and w be vertices in $G(e)$. If $\mathcal{L}(v)_e = -\mathcal{L}(w)_e$, then the definition of a cocircuit graph implies that on any path in G from v to w there is a vertex u with $\mathcal{L}(u)_e = 0$, i.e. v and w are not connected in $G(e)$. Let us assume $\mathcal{L}(v)_e = \mathcal{L}(w)_e$. The claim follows when we show that v and w are connected in $G(e)$. If $L(v) = L(w)$ then $v = w$, otherwise we apply (possibly repeatedly) Lemma 9: There exists a finite sequence of hyperplanes

$L(v) =: H_0, H_1, \ldots, H_k := L(w)$ such that $e \notin H_i$ for all $i \in \{0, \ldots k\}$ and $H_{i-1} \cap H_i =: U_i$ is a coline for all $i \in \{1, \ldots k\}$. By cocircuit axiom (C3) there exists for every $i \in \{0, \ldots, k\}$ a unique vertex v_i such that $L(v_i) = H_i$ and $\mathcal{L}(v_i)_e = \mathcal{L}(v)_e$. We show that for all $i \in \{1, \ldots, k\}$ the vertices v_{i-1} and v_i are connected in $G(e)$: Both v_{i-1} and v_i are on the coline cycle of coline U_i in G, and since $\mathcal{L}(v_{i-1})_e = \mathcal{L}(v_i)_e$ there is a (unique) path on that coline cycle from v_{i-1} to v_i in $G(e)$. ■

The property of an M-labeled cocircuit graph $G_{\mathcal{M}}$ which is stated in Proposition 8 implies that $OC(\mathcal{M})$ is uniquely determined by the M-labeled cocircuit graph, and furthermore it enables us to design a simple algorithm OMLabelFromMLabel (see Table 4) that solves the OM-labeling problem.

Input: A cocircuit graph G with M-label L.
Output: An OM-label $\mathcal{L}$ of G such that L is the M-label of G induced by $\mathcal{L}$.

begin OMLabelFromMLabel(G);
    $E := \bigcup_{v \in V(G)} L(v)$;
    for all $e \in E$ **do**
        $G(e) :=$ the subgraph of G induced by $\{v \in V(G) \mid e \notin L(v)\}$;
        let be w any vertex in $G(e)$;
        for all $v \in V(G)$ **do**

$$\mathcal{L}(v)_e := \begin{cases} 0 & \text{if } e \in L(v), \\ + & \text{if } e \notin L(v) \text{ and } v \text{ and } w \text{ are connected in } G(e), \\ - & \text{if } e \notin L(v) \text{ and } v \text{ and } w \text{ are not connected in } G(e). \end{cases}$$

        endfor
    endfor;
    return $\mathcal{L}$
end OMLabelFromMLabel.

Table 4: Algorithm OMLabelFromMLabel

Theorem 10 *Given as input a cocircuit graph G with M-label L, then the algorithm* OMLabelFromMLabel *terminates with correct output after at most $O((m+n)\ell)$ elementary arithmetic operations, where $n := |V(G)|$, $m := |E(G)|$, $\ell := |E|$, and provided that the identity check for two elements in the ground set E is possible in constant time.* ■

Corollary 11 *The reorientation class of an OM is determined by its M-labeled cocircuit graph.* ■

Corollary 12 *The isomorphism class of a uniform OM is determined by its cocircuit graph.*

Proof: The proof follows from Theorem 7 and Corollary 11. ■

6 Characterization of Cocircuit Graphs

We discuss in this section the characterization problem for cocircuit graphs of uniform OMs and of any M-labeled cocircuit graphs. We have presented in the previous sections polynomial algorithms for the corresponding M-labeling and the OM-labeling problems. These algorithms did not check the correctness of the input. In this section we add input checks to the above algorithms and use them for the design of polynomial algorithms that solve the two characterization problems mentioned above.

Remark that the algorithms for the M-labeling of cocircuit graphs of uniform OMs and for the OM-labeling of M-labeled cocircuit graphs may run into problems if their input is not correct. If such a problem is detected on run time, it will cause the algorithm to abort (we say then, the algorithm *fails*), otherwise the algorithm will terminate with some output. In neither case will the complexity of the algorithms be affected. If an algorithm fails, we know that its input was not correct, otherwise the output of the algorithm will be used to decide whether the input was correct or not.

We discuss first the algorithmic characterization of M-labeled cocircuit graphs.

Theorem 13 *Let G be a graph with label $L : V(G) \to 2^E$. There exists an algorithm which decides whether G is a cocircuit graph with M-label L or not, and this algorithm runs in time $O(n^3\ell^2)$, where $n := |V(G)|$ and $\ell := |E|$, provided that the identity check for two elements in the ground set E is possible in constant time.*

Proof: First we use the algorithm OMLabelFromMLabel in order to obtain a label $\mathcal{L}$ of G. Then we check the cocircuit axioms (C1) to (C4) for the set of all vertex labels $\mathcal{L}(v)$; if not all axioms are valid, we know that the input G and L was not a correct, i.e. we can stop and report that G is not a cocircuit graph with M-label L. If (C1) to (C4) are valid, we construct the cocircuit graph $G_{\mathcal{L}}$ of the OM defined by $\mathcal{L}$ and compare $G_{\mathcal{L}}$ with the input graph G. If G and $G_{\mathcal{L}}$ are the same (with vertices identified as they associate to the same cocircuits), then G is a cocircuit graph with M-label L, otherwise not. It remains to discuss the complexity of the above characterization algorithm; as we do not use any sophisticated data structure, our complexity result may be improved further. With $m := |E(G)|$, we have a complexity of $O((m + n)\ell)$ for OMLabelFromMLabel in order to compute $\mathcal{L}$; we check the cocircuit axioms which is trivially possible in $O(n^3\ell^2)$ elementary arithmetic steps. If all axioms are valid we construct the cocircuit graph $G_{\mathcal{L}}$ from $\mathcal{C}$ which can be done in $O(n^3\ell)$ elementary arithmetic steps as follows: The vertex set of $G_{\mathcal{L}}$ is the same as for G. For every vertex $v \in V(G_{\mathcal{L}})$ we determine in $O(n^2\ell)$ steps all adjacent vertices by first collecting all $w \in V(G_{\mathcal{L}})$ for which there is no $e \in E$ such that $\mathcal{L}(v)_e = -\mathcal{L}(w)_e \neq 0$, then taking as the adjacent vertices of v those w for which $(\mathcal{L}(v) \circ \mathcal{L}(w))^0$ is maximal among all such sets with w from the collection. The comparison of $G_{\mathcal{L}}$ and G can be done together with the construction of $G_{\mathcal{L}}$. Obviously the overall complexity is bounded by $O((m + n)\ell) + O(n^3\ell^2)$, where the later term is dominating because of $m \leq n^2$. ■

We discuss now the algorithmic characterization of unlabeled cocircuit graphs of uniform OMs.

Proposition 14 *Let G be a graph. There exists an algorithm which decides whether G is the cocircuit graph of some uniform OM $(E, \mathcal{C})$ or not, and this algorithm runs in time $O(n^3m\ell^2)$, where $n := |V(G)|$, $m := |E(G)|$, and $\ell := |E|$.*

Proof: First we use the algorithm described in Section 4 in order to obtain a label L of G and to decide whether G is a cocircuit graph with M-label L. This is possible in time $O(n^3m\ell^2)$. It remains to check whether G is the cocircuit graph of some *uniform* OM. For this we simply check whether $n = 2\binom{\ell}{r-1}$ and whether all labels $L(v)$ have cardinality $r - 1$, where r is determined from a vertex degree (see e.g. initialization of algorithm MLabelFromColineCycles). ■

7 Covector Construction Problem and Final Remarks

In the previous sections we have described polynomial algorithms for the labeling and characterization problems which enabled us to construct the set of cocircuits $\mathcal{C}$ from (M-labeled or certain unlabeled) cocircuit graphs. In this section we discuss the relation of the set $\mathcal{C}$ of cocircuits and the set $\mathcal{F}$ of all covectors of an OM, and we show how to construct $\mathcal{F}$ from $\mathcal{C}$ in polynomial time (in input and output). We conclude the section by stating some open problems.

We present in Table 5 an algorithm FacesFromCocircuits which constructs the set of covectors $\mathcal{F}$ from the set of cocircuits $\mathcal{C}$ in time $O(n\ell^2|\mathcal{F}|)$, where $n := |\mathcal{C}|$ and ℓ is the cardinality of the ground set E of the OM. We measure the complexity of the construction algorithm in sizes of input and output because the number of all covectors can be exponentially large compared to n. We remark that n is small not only compared to $|\mathcal{F}|$ but also compared to the cardinality of the *tope set* $\mathcal{T}$ of $\mathcal{C}$, that is the set of all maximal covectors in $\mathcal{F}$ w.r.t. $\preceq$ (i.e. $n \leq |\mathcal{T}|$ is valid for all OMs).

Input: The set $\mathcal{C} \subseteq \{-,0,+\}^E$ of cocircuits of some OM.
Output: The set $\mathcal{F} \subseteq \{-,0,+\}^E$ of faces of the OM defined by $\mathcal{C}$.

begin FacesFromCocircuits($\mathcal{C}$);
    $\mathcal{F} := \{0\}$; (initialize a sorted balanced tree $\mathcal{F}$, first containing the zero vector only)
    $\mathcal{F}_{\text{new}} := \{0\}$; (initialize a set $\mathcal{F}_{\text{new}}$, first containing the zero vector only)
    while $\mathcal{F}_{\text{new}} \neq \emptyset$ **do**
        take any X from $\mathcal{F}_{\text{new}}$ and remove it from $\mathcal{F}_{\text{new}}$;
        for all $Y \in \mathcal{C}$ **do**
            $Z := X \circ Y$;
            if $Z \notin \mathcal{F}$ **then** insert Z in $\mathcal{F}$ and add Z to $\mathcal{F}_{\text{new}}$ **endif**
        endfor
    endwhile;
    return $\mathcal{F}$
end FacesFromCocircuits.

Table 5: Algorithm FacesFromCocircuits

The correctness of algorithm FacesFromCocircuits is quite obvious. Remark that all faces are added to the set $\mathcal{F}_{\text{new}}$ exactly once. The compexity analysis uses the trivial fact that $|\mathcal{F}| \leq 3^\ell$, so $\log_3 |\mathcal{F}| \leq \ell$. The **while**-loop is executed for every X in $\mathcal{F}$ once, where every execution cost at most $O(n\ell^2)$ as we use a sorted balanced tree (i.e. the find and insert operations are both $O(\ell \log |F|)$, so $O(\ell^2)$). This leads to an overall complexity of $O(n\ell^2|\mathcal{F}|)$.

What are the optimal complexities of the algorithms discussed in this paper? We have presented several algorithms with polynomial complexities; nevertheless one might improve these complexities or show their optimality. Of special interest is the complexity for testing the cocircuit axioms for a given set $\mathcal{C}$ of sign vectors.

We have proved that the pairs of antipodal vertices are determined by the cocircuit graph of a uniform OM up to graph isomorphism, but it is an open question whether they are uniquely determined by the graph. We know that in the uniform case the distance between two antipodal vertices is $|E|-\text{rank}(\mathcal{M})+2$ and that there are exactly $2(\text{rank}(\mathcal{M})-1)$ edge-disjoint shortest paths between them. We do not know whether this property is enough to characterize the antipodal pairs; if it is sufficient, we can detect the negative of a cocircuit quite easily (remember that one can compute efficiently $\mathrm{rank}(\mathcal{M})$ and $|E|$ from $|V(G)|$ and $|E(G)|$). It is also an open question whether antipodal pairs are characterized as farthest pairs in G, i.e. whether the distance between two vertices v and w in G is equal to the diameter if and only if $v = \overline{w}$. It is easy to see that this is not true for non-uniform OMs.

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